GrEnFIn

Greening Energy Market and Finance

Climate risk management in finance: risk measures under model uncertainty

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Main contents

What's risk?

The term "risk" is used differently in everyday life and in literature, depending on the context:

- In colloquial language: occurrence of "unfavourable" events with adverse (economic) consequences.
- Concise Oxford English Dictionary: "hazard, a chance of bad consequences, loss or exposure to mischance".
- The standard "ISO 31000 - Risk Management" describes risk as the "effect of uncertainty on objectives".
- Keywords: decisions, uncertainty, events, consequences.

Climate risk

- The Earth's climate is changing: average temperatures rise, acute phenomena such as heat waves and floods grow in frequency and severity, and chronic phenomena, such as drought and rising sea levels, intensify.
- First fundamental question: how can climate change impact socioeconomic and financial systems across the world in the next decades?
- Second fundamental question: which actions should be tackled in order to mitigate climate change?
- Climate change risk assessment involves formal analysis of the consequences, likelihoods and responses to the impacts of climate change and the options for addressing them.
- In this lecture we will focus more on the impact of climate risk in financial systems.

Climate risk in finance

Financial institutions face today a two-sided climate risk: a physical impact risk and a policy risk .

- Many possible catastrophic events are linked to climate change: fires (California 2018, Australia 2020), hurricanes, floods, and probably also pandemics like Covid-19. These events may cause dramatic losses in different ways.
- Across the world, we see a tightening of climate policies and regulations to shift the economy away from fossil fuels. The restructuring is accelerated by the Paris Agreement, which sets clear aspirations to limit global warming to 1.5 or 2 degrees Celsius, and will affect all sectors and future investment patterns for global financial capital.

Both physical and policy risks can result in real financial impacts to companies and assets.

So: we have to measure a given risk..

- Kloman 1990: “risk management is a discipline for living with the possibility that future events may cause adverse effects”.
- Quantitative approaches to risk assessment often identify risk with the fluctuation of a value variable.
- Two kinds or approaches:
  1. One-sided approaches: only consideration of “unfavourable” deviations
  2. Two-sided approaches: consideration of both “favourable” and “unfavourable” deviations
- Examples of risk measurement related to climate risk in finance:
  - an insurance company might want to assess the risk of big losses in most exposed areas (i.e., Florida with hurricanes);
  - a bank might want to quantify its exposure to transition risk.

A first approach to risk measurement in Financial mathematics

- Let $(\Omega, \mathcal{F}, P)$ be a probability space.
- $\mathcal{X} = \{X$ random variable on $(\Omega, \mathcal{F}, P)$, with some integrability condition w.r.t. $P\}$
- X stands for the value of a financial position at the end of a given period (for example, liquidation time of positions).
- A risk measure ρ is a functional

$$\rho : \mathcal{X} \to \mathbb{R},$$

  assigning a risk $\rho(X)$ to the financial position represented by X.
- In financial applications, a rational decision maker tries to find a position $X \in \mathcal{X}$, with possibly some constraints, that minimizes $\rho(X)$.

- Risk measures are defined either in relation to the financial position X or to the loss $L = -X$.
- This difference must be taken into account in practical work and when applying results from the literature.
- In this lecture, the risk for us will be usually given in terms of financial positions.

Some examples of risk measures

In the examples below, $\mathbb{E}[\cdot]$ denotes expectation with respect to P, i.e. $\mathbb{E}[X] = \int_\Omega X dP$.

- Variance:
$$Var(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right].$$
- Normalized standard deviation:
$$\tilde{\sigma}(X) = \frac{\sqrt{Var(X)}}{\mathbb{E}[X]}.$$
Intuition: random variables with a large expected value often have a large variance or standard deviation
- Semivariance:
$$Var_+(X) = \mathbb{E}\left[\left((\mathbb{E}[X] - X)^+\right)^2\right].$$
Note: only shortfalls $X < \mathbb{E}[X]$ are taken into account.
- Value at Risk at level $\alpha \in (0, 1)$ of a financial position X:
$$VaR_\alpha(X) := \inf\{m \in \mathbb{R} : P(X + m < 0) \leq \alpha\}.$$
Interpretation: smallest amount of money ("risk capital") that must be added to X so that the probability of bankruptcy is $\leq \alpha$.

But: are these risk measures appropriate for climate risk?

- Problem: the climate change case illustrates particularly well a situation in which the probabilistic model, i.e., the probability measure P, is neither explicitly given nor can it be adequately approximated or inferred with the available data and current scientific methods: deep model uncertainty.
- These uncertainties arise from both the extreme complexity of the climatic system and our inability to perfectly capture the way our socioeconomic system would respond and adapt to climate change.
- This is particularly the case when we consider situations with potential catastrophic consequences, such as the collapse of the Atlantic thermohaline circulation, the melting of the Antarctic ice sheet or the loss of the Amazon rainforest. Such catastrophic events (also called tipping points) have not been encountered in recent history, and therefore their likelihood of occurrence is extremely difficult to assess.

How to deal with this issue?

- In view of this disagreement among experts or models, how should a rational policy decision maker proceed?
- If one follows the traditional subjective risk minimazation approach, one would simply aggregate the models by averaging them into a single representative model.
- The problem with this approach is that the decision maker considers the resulting aggregated model in exactly the same way as one would consider an equivalent objective model representing a specific risk, and model uncertainty has therefore no impact on the decision-making process.
- Ellsberg (1961) showed through different experiments that the choices of individuals cannot be rationalized under the traditional Bayesian expected utility paradigm, and that individuals usually manifest aversion toward situations in which probabilities are not perfectly known.

Let's start with an example: Ellsberg paradox

Urn with 90 balls: 30 red, 60 black *OR* white.
People have been required to answer the following questions:

1. do you prefer to receive $100\$$ when you:
   a. draw a red ball
   b. draw a white ball
2. do you prefer to receive $100\$$ when you:
   a. draw a red or black ball
   b. draw a white or black ball

Try to guess the most common answers..

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Try to guess the most common answers..

- (a) to point 1, (b) to point 2.
- But why? Relying on utility theory, if you prefer red to white you also prefer [red or black] to [white or black]!
- Possible reason: people are averse to model uncertainty.
- Let's go more into details..

Risk vs uncertainty (Knight, 1921)

- There's a difference between two types of "imperfect knowledge":
  1. risk (or measurable uncertainty) $\rightarrow$ situations in which the distribution of the target random variables is known;
  2. Knightian, model, or not measurable) uncertainty $\rightarrow$ the distribution of the target random variables is not known. This is the case for many issues related to climate risk.
- Think about the previous example: if you win when you draw a red ball, your gamble is based on a distribution you know: $P(\text{win}) = \frac{1}{3}$. This is not the case if you win when the white ball is drawn. Same thing for the second choice.
- The example shows that people do no treat these kinds of uncertainty in the same way: ambiguity (or model uncertainty) aversion.

- Standard procedure: modelling under the usual concept of “Risk”:
  - Tacit assumption: a fixed probability measure P, and thus the distribution of the underlying random variables/sources of risk, is known.
  - Example in financial mathematics: we specify the dynamics of some stochastic processes with respect to a fixed probability P and we price derivatives based on those dynamics.
- The assumption above is not realistic for climate risk (as well as in other fields of finance).
- Approach under model uncertainty: probabilities are unknown for financial market events $\rightarrow$ Increased awareness of the problems that can result from excessive reliance on a specific probabilistic model is needed.

- Instead of a reference measure P, consider a family $\mathcal{P}$ of possible probability measures. Each element of $\mathcal{P}$ reflects a possible different model, which gives rise to a different probability distribution.
- Example: utility maximization under model uncertainty
  - S stochastic process with log-normal returns R_k, i.e.,

$$S_T = S_0 e^{R_1+R_2+\cdots+R_T}.$$

  - Introduce a family of probability measures to express uncertainty about returns:

$$\mathcal{P} := \{P^\mu | \mu \in [a,b] \text{ and } R_1, R_2, \ldots, R_T \text{ i.i.d. }, R_k \sim \mathcal{N}(\mu, \sigma^2) \text{ under } P^\mu\}.$$

  - The maximization of the expected utility of a financial position X involving S and a risk-free asset can be achieved by

$$\text{maximize} \inf_{P \in \tilde{P}} \mathbb{E}^{\tilde{P}}[u(X)], \quad X \in \mathcal{X},$$

    $u(\cdot)$ utility function, $\mathcal{X}$ family of financial positions: maxmin approach.

Model uncertainty and risk management

- There are different possible ways to deal with model uncertainty in risk management (and so in particular in the setting of climate risk).
- A key idea is that risk measures should be robust with respect to model uncertainty.
- Accordingly, risk minimization should take model uncertainty into account.
- In the next slides, we will see how to define robust risk measures and how to include model uncertainty into a risk minimization problem.
- The next section is based on the paper Robust Preferences and Convex Measures of Risk, Föllmer and Schied, 2002.

Motivation

- At the turn of the millennium, the weaknesses of Value at Risk led to the development of an axiomatic theory of risk measures:
  - P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, Coherent measures of risk, Mathematical Finance 9, 1999;
  - H. Föllmer, A. Schied, Convex measures of risk and trading constraints. Finance $\&$ Stochastics 4, 2000.
- Core ideas:
  1. The risk of a position X has to be quantified as the minimum capital which must be added to X so that the position becomes acceptable (e.g. from the point of view of a supervisory authority)
  2. Diversification must be incentivated: subadditivity/convexity

The setting

- Take a measurable space $(\Omega, \mathcal{F})$, standing for possible scenarios. Note: no probability measure is specified!
- A financial position is modelled by a random variable $X : \Omega \to \mathbb{R}$: $X(\omega)$ is the discounted value of the position at the end of a given period (liquidation time, as before) in the scenario ω.
- The space $\mathcal{X}$ of all possible positions is a linear subspace of measurable functions on $(\Omega, \mathcal{F})$, which contains the constants.

A functional $\rho : \mathcal{X} \to \mathbb{R}$ is called monetary risk measure if it satisfies the following properties:

1. no position is "infinitely good": $\rho(X) > -\infty$ for all $X \in \mathcal{X}$;
2. for every constant $m \in \mathbb{R}$ it holds $\rho(m) < +\infty$;
3. monotonicity: if $X \leq Y$ (i.e., $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$), it holds $\rho(X) \geq \rho(Y)$;
4. cash invariance: for every $m \in \mathbb{R}$ it holds $\rho(X + m) = \rho(X) - m$: if a capital m is added to a position X, the risk of new position $X + m$ is reduced by amount m.

Acceptance sets

A set $\mathcal{A} \subset \mathcal{X}$ is said to be an acceptance set if:

1. $\mathcal{A} \cap \{$constant functions$\} \neq \emptyset$: $\exists m \in \mathbb{R}$ such that having m is acceptable;
2. For all $X \in \mathcal{X}$ there exists $m \in \mathbb{R}$ such that $X + m \neq \mathcal{A}$: no position is “infinitely good”;
3. $\mathcal{A}$ is monotone in the sense that $X \in \mathcal{A}$, $Y \in \mathcal{X}$ and $Y \geq X$ implies $Y \in \mathcal{A}$.

Proposition

Let $\mathcal{A} \subset \mathcal{X}$ be an acceptance set. Thus the functional $\rho_{\mathcal{A}} : \mathcal{X} \to \mathbb{R}$ defined by

$$\rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}$$

is a monetary risk measure.

Proposition

Let a functional $\rho : \mathcal{X} \to \mathbb{R}$ be a monetary risk measure. Thus the set $\mathcal{A}_\rho$ defined by

$$\mathcal{A}_\rho := \{X \in \mathcal{X} : \rho(X) \leq 0\}$$

is an acceptance set.

Convex risk measures

Remember: diversification should not increase risk.

Definition

A monetary risk measure ρ is called a convex risk measure if for every $\lambda \in [0, 1]$, $X, Y \in \mathcal{X}$ it holds

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y).$$

Proposition

A monetary risk measure is convex if and only if for every $\lambda \in [0, 1]$, $X, Y \in \mathcal{X}$ it holds

$$\rho(\lambda X + (1-\lambda)Y) \leq \max(\rho(X), \rho(Y)).$$

Proposition

A monetary risk measure ρ is convex if and only if $\mathcal{A}_\rho$ is a convex set.

Coherent risk measures and subadditivity

Definition

A convex risk measure ρ is called coherent risk measure if it is positive homogenous, i.e., if for every $\lambda \geq 0$, $X \in \mathcal{X}$ it holds

$$\rho(\lambda X) = \lambda \rho(X).$$

Proposition

A coherent risk measure is subadditive, i.e., for every $X, Y \in \mathcal{X}$ it holds

$$\rho(X+Y) \leq \rho(X) + \rho(Y).$$

Proposition

A monetary risk measure ρ is coherent if and only if $\mathcal{A}_\rho$ is a convex cone.

Remark

Coherent risk measures have been criticized since the positive homegeneity property does not always hold in practice, for example due to illiquidity effects. See among others:

- H. Föllmer and A. Schied, *Convex measures of risk and trading constraints*. Finance and Stochastics, 2002, 6.4: 429-447
- M. Frittelli, and E. Rosazza Gianin, *Putting Order in Risk Measures*, J.Bank. Finance, 2002, 26, 1473-1486

What about model uncertainty?

- Note that for now we have not fixed any probability measure, so no model for our risky financial position X.
- On the other hand, no notions of robustness with respect to model uncertainty have been specified.
- This is what we want to do now.

Definition

A risk measure ρ admits a robust representation if for every $X \in \mathcal{X}$ it holds

$$\rho(X) = \sup_{Q \in \mathcal{M}} \left\{ \mathbb{E}^Q[-X] - \alpha(Q) \right\},$$

where

$$\mathcal{M} = \{\text{probability measures } Q \text{ on } (\Omega, \mathcal{F}) \text{ such that } \mathbb{E}^Q[X] \text{ is finite for every } X \in \mathcal{X}\}.$$

The functional $\alpha : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$ is called penalty function.

Interpretation

- The elements of $\mathcal{M}$ can be interpreted as possible probabilistic models, which are taken more or less “seriously” according to the size of the penalty $\alpha(Q)$.
- The value $\rho(X)$ is computed as the worst case expectation taken over all models $Q \in \mathcal{M}$ and penalized by $\alpha(Q)$.

Robust representation for coherent measures

Proposition

A risk measure ρ satisfying the representation above is convex.

Proof

Let $\lambda \in (0,1)$, and suppose that ρ has the representation

$$\rho(X) = \sup_{Q \in \mathcal{M}} \left\{ \mathbb{E}^Q[-X] - \alpha(Q) \right\},$$

with $\alpha : \mathcal{M} \to \mathbb{R}^+ \cup \{+\infty\}$. Then for every $X, Y \in \mathcal{X}$ and $\lambda \in (0,1)$ it holds

$$\begin{aligned}
\rho(\lambda X + (1-\lambda)Y) &= \sup_{Q \in \mathcal{M}} \left\{ \mathbb{E}^Q[-\lambda X - (1-\lambda)Y] - \alpha(Q) \right\} \\
&= \sup_{Q \in \mathcal{M}} \left\{ \lambda \mathbb{E}^Q[-X] + (1-\lambda)\mathbb{E}^Q[-Y] - \lambda\alpha(Q) - (1-\lambda)\alpha(Q) \right\} \\
&= \sup_{Q \in \mathcal{M}} \left\{ \lambda \left(\mathbb{E}^Q[-X] - \alpha(Q) \right) + (1-\lambda) \left(\mathbb{E}^Q[-Y] - \alpha(Q) \right) \right\} \\
&\leq \lambda \sup_{Q \in \mathcal{M}} \left\{ \mathbb{E}^Q[-X] - \alpha(Q) \right\} + (1-\lambda) \sup_{Q \in \mathcal{M}} \left\{ \mathbb{E}^Q[-Y] - \alpha(Q) \right\} \\
&= \lambda\rho(X) + (1-\lambda)\rho(Y).
\end{aligned}$$

Proposition

- A risk measure ρ which admits a robust representation is coherent if and only if the penalty function α only takes the values 0 and ∞, i.e.

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q[-X]$$

where $\mathcal{Q} = \{Q \in \mathcal{M} : \alpha(Q) = 0\}$.

Existence of a robust representation for convex risk measures

- We assume that $\mathcal{X}$ is the linear space of all bounded measurable functions on a measurable space $(\Omega, \mathcal{F})$.
- For this reason, $\mathcal{M}$ is now the set of all probability measures on $(\Omega, \mathcal{F})$.
- We assume that Ω is a Polish space, i.e., a separable topological space admitting a complete metric.
- We also suppose $\mathcal{F}$ to be the Borel σ-algebra.
- We denote by $C_b(\Omega)$ the subspace of X of bounded continuous functions on Ω.

Definition

A convex risk measure ρ on $\mathcal{X}$ is called tight if there exists an increasing sequence $K_1 \subset K_2 \subset \cdots$ of compact subsets of Ω such that

$$\rho(\lambda \mathbf{1}_{\{K_n\}}) \searrow \rho(\lambda) \quad \text{for all } \lambda \geq 1.$$

Existence of a robust representation for convex risk measures

Theorem

Let ρ be a convex risk measure on $\mathcal{X}$. Then the following conditions are equivalent:

1. (i) ρ is tight;
2. (ii) ρ is continuous from below in $C_b(\Omega)$, i.e., if $(X_n)_{n\in\mathbb{N}}$ is a sequence in $C_b(\Omega)$ such that $X_n \nearrow X \in C_b(\Omega)$, then $\rho(X_n) \searrow \rho(X)$.

If one of the two conditions above is satisfied, ρ has the robust representation

$$\rho(X) = \sup_{Q\in\mathcal{M}} \left(\mathbb{E}^Q[-X] - \alpha(Q)\right)$$

for a given penalty functional α.
If ρ is coherent and one of the two conditions above is satisfied, ρ has the robust representation

$$\rho(X) = \sup_{Q\in\mathcal{Q}} \mathbb{E}^Q[-X],$$

for a given subset $\mathcal{Q} \subseteq \mathcal{M}$.

Remark

In the following examples a probability measure P is fixed in $(\Omega, \mathcal{F})$ and the linear space $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, P)$ is considered. All risk measures are initially defined on $\mathcal{X}$, but have canonical extensions to larger spaces.
The expectation will be always taken with respect to P unless differently specified, i.e.

$$\mathbb{E}[X] = \int_\Omega X dP.$$

Note that since we consider bounded random variables, the set $\mathcal{M}$ introduced above is the space of probability measures in $(\Omega, \mathcal{F})$.

Value at Risk: no convex, no robust representation!

- The Value at Risk at level λ is a monetary risk measure with acceptance set

$$\mathcal{A}_\lambda = \{X \in \mathcal{X} : P(X < 0) \leq \lambda\}.$$

- In terms of capital requirement:

$$\begin{aligned} VaR_\lambda(X) &= \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_\lambda\} \\ &= \inf\{m \in \mathbb{R} : P(X + m < 0) \leq \lambda\}. \end{aligned}$$

- Note: Value at Risk is a positive homogenous monetary measure, but not convex!
- It follows that not only Value at Risk does not reward diversification, but from the proposition we have seen it also fails to have a robust representation.

Average Value at Risk/Expected Shortfall

Definition

The Average Value at Risk at level $\lambda \in (0,1]$ for a position X is

$$AVaR_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda VaR_\beta(X)d\beta.$$

- As opposed to Value at Risk, it takes into account extreme losses.
- Since $\lambda \to VaR_\lambda$ is non-decreasing, it holds

$$AVaR_\lambda(X) \geq VaR_\lambda(X):$$

  Average Value at Risk is more conservative with respect to Value at Risk.
- It is a coherent risk measure with robust representation

$$AVaR_\lambda(X) = \sup_{Q \in \mathcal{Q}_\lambda(P)} \mathbb{E}^Q[-X]$$

  with

$$\mathcal{Q}_\lambda(P) := \left\{ Q \in \mathcal{M}, Q \ll P : \frac{dQ}{dP} \leq \frac{1}{\lambda} \right\}.$$

Utility-based shortfall risk

Definition

A loss function is a convex, increasing and twice differentiable function $\ell : \mathbb{R} \to \mathbb{R}$.

Definition

Consider a loss function ℓ and take $r_0 > \inf_{x \in \mathbb{R}}\{\ell(x)\}$. The utility-based shortfall risk ρ for a position $X \in \mathcal{X}$ is defined as

$$\rho(X) = \inf\{m : X + m \in \mathcal{A}\},$$

where

$$\mathcal{A} := \{X \in \mathcal{X} : \mathbb{E}[\ell(-X)] \leq r_0\}.$$

Remark

Introduce the utility function $u(x) := r_0 - \ell(-x)$. We can write the acceptance set $\mathcal{A}$ as

$$\mathcal{A} := \{X \in \mathcal{X} : \mathbb{E}[u(X)] \geq 0\}.$$

This is why ρ is called “utility-based shortfall risk”.

More on the utility-based shortfall risk

- The acceptance set $\mathcal{A}$ is convex, so ρ is a convex risk measure.
- If X has a continuous distribution and if ℓ is continuous, $m = \rho(X)$ is the unique solution to the equation

$$\mathbb{E}[\ell(-X-m)] = r_0.$$

  This can be solved by numerical methods.
- The risk measure ρ admits a robust representation

$$\rho(X) = \sup_{Q \in \mathcal{M}_1(P)} \{\mathbb{E}^Q[-X] - \alpha(Q)\},$$

  where $\mathcal{M}_1(P) = \{Q \in \mathcal{M}, Q \ll P\}$ and

$$\alpha(Q) = \inf_{\lambda > 0} \frac{1}{\lambda} \left(r_0 + \mathbb{E}\left[\ell^* \left(\lambda \frac{dQ}{dP} \right) \right] \right),$$

  where $\ell^*(z) := \sup_{x \in \mathbb{R}}\{zx - \ell(x)\}$ is the Fenchel-Legendre transform.

Entropic risk measure

Definition

For a fixed probability measure P and a parameter $\gamma > 0$, the entropy penalty function is defined as $\alpha(Q) := \frac{1}{\gamma} H(Q|P)$, where

$$H(Q|P) := \begin{cases} \mathbb{E}^Q\left[\ln \frac{dQ}{dP}\right] & \text{if } Q \ll P \\ +\infty & \text{otherwise} \end{cases}$$

Interpretation: the more a measure Q "diverges" from P, the more it gets penalized.

Definition

For a fixed probability measure P and a parameter $\gamma > 0$, the entropic risk measure for a position X is defined by the robust representation with respect to the entropy penalization function defined above:

$$e_\gamma(X) := \sup_{Q \in \mathcal{M}} \{\mathbb{E}^Q[-X] - \alpha(Q)\}.$$

More on the entropic risk measure

- It can be seen that

$$H(Q|P) = \sup_{X \in L^\infty(\Omega,\mathcal{F},P)} \{\mathbb{E}^Q[-X] - \ln \mathbb{E}[e^{-X}]\}.$$

- It follows the explicit representation

$$e_\gamma(X) = \frac{1}{\gamma} \ln \mathbb{E}[e^{-\gamma X}]. \tag{1}$$

- Define the loss function $\ell(x) = e^{\gamma x}$ and the utility function $u(x) = 1 - e^{-\gamma x}$. Thus it holds

$$\mathcal{A} = \{X \in \mathcal{X} | e_\gamma(X) \le 0\} = \{X \in \mathcal{X} | \mathbb{E}[\ell(-X)] \le 1\} = \{X \in \mathcal{X} | \mathbb{E}[u(X)] \ge 0\}.$$

- Then, e_γ is a special case of the utility-based shortfall risk measure.
- Also note that (1) can be written as

$$e_\gamma(X) = \ell^{-1}\left(\mathbb{E}\left[\ell(-X)\right]\right)$$

for $\ell(x) = e^{\gamma x}$. This is known as certainty equivalent of ℓ.

Certainty equivalents

- Consider a probability space $(\Omega, \mathcal{F}, P)$, and the space $\mathcal{X} := L^\infty(\Omega, \mathcal{F}, P)$ of bounded random variables in such a probability space.
- We have seen that the entropic risk measure e_γ can be written as

$$e_\gamma(X) = \ell^{-1}\left(\mathbb{E}\left[\ell(-X)\right]\right)$$

  for the loss function $\ell(x) = e^{\gamma x}$.
- The expression above is the certainty equivalent for that specific loss function ℓ.
- Let us now follow a more general approach to certainty equivalents. Take a given loss function ℓ, i.e., a convex, increasing, twice differentiable function $\ell : \mathbb{R} \to \mathbb{R}$, and define the certainty equivalent ρ^ℓ of ℓ as

$$\rho^\ell(X) := \ell^{-1}\left(\mathbb{E}\left[\ell(-X)\right]\right).$$

- In terms of the utility function $u_\ell(x) = -\ell(-x)$,we can write ρ^ℓ as

$$\rho^\ell(X) := -u_\ell^{-1}\left(\mathbb{E}\left[u_\ell(X)\right]\right).$$

Definition of Quasi-linearity

A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is called quasi-linear if $\rho(X) = \rho(Y)$ implies

$$\rho\left(\alpha X + (1-\alpha)Y\right) = \rho\left(\alpha Y + (1-\alpha)X\right)$$

for all $X, Y \in \mathcal{X}$ and for all $\alpha \in (0,1)$.

The following is a well known theorem which characterizes certainty equivalents.

Nagumo-Kolmogorov-de Finetti Theorem

Any functional $\rho : \mathcal{X} \to \mathbb{R}$ can be written in the form

$$\rho(X) := \ell^{-1}\left(\mathbb{E}\left[\ell(-X)\right]\right)$$

for some loss function $\ell : \mathbb{R} \to \mathbb{R}$ if and only if it satisfies the following properties:

- monotonicity;
- law invariance: $\rho(X) = \rho(Y)$ if $X \sim Y$;
- constancy: $\rho(m) = -m$ for any $m \in \mathbb{R}$;
- quasi-linearity.

Optimization problems

- Certainty equivalents are quite often used in order to define optimization problems. One typically seeks to solve the minimization problem

$$\min_{X \in \bar{\mathcal{X}}} \rho^{\ell}(X),$$

where $\bar{\mathcal{X}}$ is a given (admissible) subset of $\mathcal{X}$, or

$$\min_{a \in \mathcal{A}} \rho^{\ell}(-X(a)), \tag{2}$$

where $\mathcal{A}$ is a set of possible actions and where we denote by $X(a)$ a random variable depending on the action $a \in \mathcal{A}$, i.e., $X(a, \omega)$ for $\omega \in \Omega$.

- Intuitively, one wants to find an optimal position, or an optimal action, that minimizes her/his risk measure, which is seen here as depending on the expectation of future losses.
- In our application to climate risk, we will consider an optimal emission abatement problem which is close to (2) in its form.
- For this reason, we focus more on problem (2) in the next section.

Risk minimization under model uncertainty

- In the previous section, we have fixed a probability measure. That is, if one wants to solve the problem

$$\min_{a\in\mathcal{A}} \rho^{\ell}(X(a)) = \min_{a\in\mathcal{A}} \ell^{-1}\left(\mathbb{E}\left[\ell(-X(a))\right]\right),$$

  still does this referring to the corresponding probability measure.
- First idea: for some specific loss functions ℓ, the associated certainty equivalent ρ^{ℓ} is convex, and has a robust representation. So we are indirectly taking into consideration also other measures.
- Second idea: we could try to directly incorporate model uncertainty into the optimization problem.
- That is, directly include the set of possible probability measures into the problem.

Decision forms and model uncertainty

- As before, we identify uncertainty with a set $\mathcal{P}$ of possible probability measures on a space $(\Omega, \mathcal{F})$.
- In the climate risk setting, the set $\mathcal{P}$ may represent for example all the possible probabilities for a catastrophic event such as the collapse of the Atlantic meridional overturning circulation.
- We introduce a subjective prior probability μ defined on the power set of $\mathcal{P}$.
- Following the approach of S. Cerreia-Vioglio et al. [1], one can define the P-certainty equivalent

$$\rho^{P,\ell}(X) := \ell^{-1}\left(\mathbb{E}^{P}\left[\ell(-X)\right]\right)$$

for any $P \in \mathcal{P}$, and embed problem (2) into the setting of model uncertainty by defining the new risk-minimization problem

$$\min_{a\in\mathcal{A}} \int_{\mathcal{P}} \ell\left(\rho^{P,\ell}(X(a))\right) d\mu(P).$$

[1] F. Maccheroni, M. Marinacci, and L. Montrucchio, *Classical Subjective Expected Utility.*, Proceedings of the National Academy of Sciences, 110, 6754-6759

Induced predictive probability

Note that

$$\begin{aligned}\min_{a\in\mathcal{A}}\int_{\mathcal{P}}\ell\left(\rho^{P,\ell}(X(a))\right)d\mu(P) &= \min_{a\in\mathcal{A}}\int_{\mathcal{P}}\ell\left(\ell^{-1}\left(\mathbb{E}^{P}\left[\ell(-X(a))\right]\right)\right)d\mu(P).\\ &= \min_{a\in\mathcal{A}}\int_{\mathcal{P}}\mathbb{E}^{P}\left[\ell(-X(a))\right]d\mu(P)\\ &= \min_{a\in\mathcal{A}}\int_{\mathcal{P}}\left(\int_{\Omega}\ell\left(-X(a,\omega)\right)dP(\omega)\right)d\mu(P)\\ &= \min_{a\in\mathcal{A}}\int_{\Omega}\ell\left(-X(a,\omega)\right)d\bar{\mu}(\omega),\end{aligned}$$

where the predictive probability $\bar{\mu}\in\mathcal{P}$ is defined by

$$\bar{\mu}(B)=\int_{\mathcal{P}}P(B)d\mu(P)$$

for $B\in\mathcal{F}$, so that (formally)

$$d\bar{\mu}(\omega)=\bar{\mu}(d\omega)=\int_{\mathcal{P}}P(d\omega)d\mu(P)=\int_{\mathcal{P}}dP(\omega)d\mu(P).$$

Back to the classical case?

- We have seen that each prior $\mu \in \mathcal{P}$ induces a predictive probability $\bar{\mu} \in \mathcal{P}$, by

$$\bar{\mu}(B) = \int_{\mathcal{P}} P(B) d\mu(P)$$

for $B \in \mathcal{F}$, such that

$$\min_{a \in \mathcal{A}} \int_{\mathcal{P}} \ell \left(\rho^{P,\ell}(X(a)) \right) d\mu(P) = \min_{a \in \mathcal{A}} \int_{\Omega} \ell \left(-X(a, \omega) \right) d\bar{\mu}(\omega).$$

- But then, modulo a change of probability measure, we are back to the classical setting!
- Such a probability measure $\bar{\mu}$ basically aggregates all the different probability measures, i.e., all the different models.

Example: the first question of the Ellsberg paradox

- Remember: we have an urn with 90 balls: 30 red, 60 black *OR* white.
- We compare the *expected utility* of the two actions bet on red and bet on white.
- Clearly, the probability to draw a red ball is $P(\text{red}) = \frac{1}{3}$.
- About the white ball, we have model uncertainty. In particular,

$$\mathcal{P} = \left\{ P_i, \quad P_i(\text{white}) = \frac{i}{90}, \quad i = 0, 1, \dots, 60 \right\}.$$

- Suppose that every probability measure $P_i \in \mathcal{P}$ has subjective prior probability $\frac{1}{61}$:

$$\mu(P_i) = \frac{1}{61}, \qquad i = 0, \dots, 60.$$

- What about $\bar{\mu}$ then? We have

$$\bar{\mu}(\text{white}) = \sum_{i=0}^{60} P_i(\text{white})\,\mu(P_i) = \frac{1}{61}\,\frac{0+1+\cdots+59+60}{90} = \frac{1}{61}\,\frac{60 \cdot 61}{2}\,\frac{1}{90} = \frac{1}{3}.$$

- So we should be neutral between betting on red or white. Evidence shows that this is not the case. Again: we have to include ambiguity aversion! How?

How to include ambiguity aversion in our formulation?

- Remember that we formalized our risk minimization problem under model uncertainty as

$$\min_{a\in\mathcal{A}} \int_{\mathcal{P}} \ell\left(\rho^{P,\ell}(X(a))\right) d\mu(P) = \min_{a\in\mathcal{A}} \int_{\mathcal{P}} \mathbb{E}^P\left[\ell(-X(a))\right] d\mu(P).$$

- The simplification above makes this approach easily tractable.
- But: basically we are using the same loss function ℓ (which we take convex in order to represent our aversion to risk) also in order to deal with model uncertainty!
- Doing so, we don't really account for ambiguity aversion.
- We want then to generalize the representation above by distinguishing such an attitude: we introduce another loss function $\tilde{\ell} : \mathbb{R} \to \mathbb{R}$ and consider the problem

$$\min_{a\in\mathcal{A}} \int_{\mathcal{P}} \tilde{\ell}\left(\rho^{P,\ell}(X(a))\right) d\mu(P) = \min_{a\in\mathcal{A}} \int_{\mathcal{P}} \tilde{\ell}\circ\ell^{-1}\left(\mathbb{E}^P\left[\ell(-X(a))\right]\right) d\mu(P)$$

- Accordingly, we can introduce the ambiguity aversion-certainty equivalent

$$\rho^{\mathcal{P},\tilde{\ell}}(X) = \tilde{\ell}^{-1}\left(\int_{\mathcal{P}} \tilde{\ell}\left(\rho^{P,\ell}(X(a))\right) d\mu(P)\right) = \tilde{\ell}^{-1}\left(\int_{\mathcal{P}} \tilde{\ell}\circ\ell^{-1}\left(\mathbb{E}^P\left[\ell(-X(a))\right]\right) d\mu(P)\right)$$

  and define the optimization problem

$$\min_{a\in\mathcal{A}} \rho^{\mathcal{P},\tilde{\ell}}(X(a)).$$

- Ambiguity aversion corresponds to the convexity of the function $\phi := \tilde{\ell}\circ\ell^{-1}$.
- In particular, ϕ is convex if and only if $\tilde{\ell}$ is more convex than ℓ - that is, if there exists a strictly increasing and convex function g such that $\tilde{\ell} = g\circ\ell$.

A model for abatement's level decisions

- This section is based on the paper *Managing catastrophic climate risks under model uncertainty aversion*, L. Berger, J. Emmerling, M Tavoni, Management Science 63.3 (2017): 749-765.
- The focus here is on emission abatement decisions taken from a decision maker.
- We have two periods, today and the future. During the first period, the DM chooses a level of emission abatement a which has (present) cost $c(a)$. This erodes the present wealth w_1, which is therefore $w_1 - c(a)$.
- In the future, there are two possible categories of states of the world. One is safe: under this scenario, a (deterministic) wealth w_2 is provided in the future.
- In the second scenario, the environment is severely affected by a catastrophic event (example: the collapse of the Atlantic meridional overturning circulation) that gives rise to other unfavourable accidents in a set S. An event $s \in S$ causes a damage L_s, such that the future wealth becomes $w_2 - L_s$, and occurs with probability π_s, conditional on the scenario taking place.
- The probability that such a catastrophic event will occur depends on the level of abatement a chosen in the first period.

The introduction of model uncertainty: the probability of the catastrophic event is unknown

- Model uncertainty is introduced assuming that the probability of the catastrophic event (as a function of a) is not known.
- In particular, we introduce n models $\theta \in \{1, \dots, n\}$ and define the set

$$\mathcal{P} = \{P_\theta(\cdot)\}_{\theta \in \{1,\dots,n\}},$$

  where $P_\theta(a)$ is the probability of the catastrophic event according to the model $\theta \in \{1, \dots, n\}$ if the abatement level a is chosen.
- We suppose that the DM does not know which of the probabilities P_θ is the true or the most accurate one, but can associate a subjective prior probability $\mu(\theta)$, in the notation of the previous section, to all of them.
- In particular, we suppose that the DM gives subjective prior probability $0 \le q_i \le 1$ to the model $\theta = i$, for every $i = 1, \dots, n$, where

$$\sum_{i=1}^{n} q_i = 1.$$

The risk minimization problem under model uncertainty

- The problem is choosing the optimal abatement level a taking into account the cost $c(a)$ on one side, and the probabilities $P_\theta(a)$, $\theta \in \{1, \dots, n\}$, on the other side.
- We can follow the approach of last section, with a slight difference: here the action $a \in \mathcal{A}$ does not affect directly the future wealth $w_2 - L_s$, $s \in S$, but the probability that such losses occur.
- Let L be the loss random variable, i.e., the random variable L such that $L = L_s$ with probability π_s in the catastrophic scenario and $L = 0$ in the other one.
- We can formulate the problem as

$$\min_{a \in \mathcal{A}} \left(\ell(c(a) - w_1) + \beta \int_{\mathcal{P}} \tilde{\ell} \left(\rho^{P_\theta, a, \ell}(w_2 - L) \right) d\mu(P_\theta) \right),$$

where $\beta \in (0, 1]$ is the discount factor of future costs and

$$\rho^{P_\theta, a, \ell}(w_2 - L) := \ell^{-1} \left(\mathbb{E}^{P_\theta, a}[\ell(L - w_2)] \right)$$

is a certainty equivalent taken with respect to the expectation under the model P_θ, for some $\theta \in \{1, \dots, n\}$, if the abatement level $a \in \mathcal{A}$ is chosen.

- Note that we have
$$\mathbb{E}^{P_\theta,a}[\ell(L-w_2)] = P_\theta(a)\sum_{s\in S}\ell(L_s-w_2) + (1-P_\theta(a))\ell(-w_2).$$
- Therefore,
$$\int_{\mathcal{P}} \tilde{\ell}\left(\rho^{P_\theta,a,\ell}(w_2-L)\right)d\mu(P_\theta) = \sum_{i=1}^{n} q_i\phi\left(\mathbb{E}^{P_{\theta_i},a}[\ell(L-w_2)]\right)$$
$$= \sum_{i=1}^{n} q_i\phi\left(P_{\theta_i}(a)\sum_{s\in S}\ell(L_s-w_2) + (1-P_{\theta_i}(a))\ell(-w_2)\right),$$
with $\phi = \tilde{\ell}\circ\ell^{-1}$.
- Thus, we can write the optimization problem under model uncertainty as
$$\min_{a\in\mathcal{A}}\left(\ell(c(a)-w_1) + \beta\sum_{i=1}^{n} q_i\phi\left(P_{\theta_i}(a)\sum_{s\in S}\ell(L_s-w_2) + (1-P_{\theta_i}(a))\ell(-w_2)\right)\right),$$
with $\phi = \tilde{\ell}\circ\ell^{-1}$.
- As before, model uncertainty aversion corresponds to the convexity of ϕ.

The following definition is taken word by word from the paper *Precautionary saving and the notion of ambiguity prudence*, L. Berger, Economics letters 123.2 (2014): 248-251, where such a concept has been first formally introduced.

Definition

An agent is ambiguity prudent if the introduction of ambiguity through a mean-preserving spread in the space of first order distributions of his future wealth raises his optimal level of saving.

Note that ambiguity prudence differs from ambiguity aversion: it only refers to the amount of money that an agent is willing to invest under or without ambiguity.

The degree of model disagreement

Definition

For any set of probability functions $P_\theta(\cdot)_{\theta \in \{1,2,\dots,n\}}$, the degree of model disagreement is defined by

$$\sigma^2(a) := \text{Var}_\theta[P_\theta(a)]$$

for any given level of abatement a.

Proposition

The degree of model disagreement $\sigma^2(a)$ is decreasing (increasing) in abatement if and only if, for any given level of abatement a, it holds

$$\text{Cov}\left(P_\theta(a), \frac{\partial P_\theta(a)}{\partial a}\right) \leq (\geq) 0.$$

Intuitively, the above proposition can be understood as follows:

- $\text{Cov}\left(P_\theta(a), \frac{\partial P_\theta(a)}{\partial a}\right) \leq 0$ means that the higher the probability $P_\theta(a)$, the smaller the (negative!) derivative $\frac{\partial P_\theta(a)}{\partial a}$.
- That is, the most pessimistic models are also the ones for which the probability of the catastrophic event decreases faster in the level of abatement.
- That is, such probabilities converge for high levels of abatement.

Main result

Definition

We call a classical subjective expected loss (CSEL) maximizer a DM that does not show ambiguity aversion, i.e., whose preferences are represented by a function $\tilde{\ell} = \ell$.

Proposition

A DM exhibiting (strict) ambiguity prudence and ambiguity aversion always chooses to abate (strictly) more than a CSEL maximizer if the degree of model disagreement decreases or is constant in abatement.

- The above proposition says that if higher abatement leads to a reduction in the degree of model disagreement, a positive incentive is generated to abate more, under ambiguity aversion and ambiguity prudence.
- From what we have observed before, the degree of model disagreement decreases in abatement if abatement decreases the probability of a catastrophe more strongly in more pessimistic models.

The experiment

- The study of Zickfeld et al.[2] presents the results from interviews with 12 leading climate scientists about the risk of a collapse of the Atlantic meridional overturning circulation (AMOC) due to global warming.
- Specifically, the authors collected the experts' probabilities that a collapse of the AMOC will occur or will be irreversibly triggered as a function of global mean temperature increase realized by the year 2100.
- Such probabilities are defined as functions of the change in global mean temperature. Of course, the higher the change of the temperature, the higher the probability.
- In our model, the change in global mean temperature is a decreasing function of the abatement.[3] That is, the degree of model disagreement decreases under abatement if it increases under the change of the temperature.

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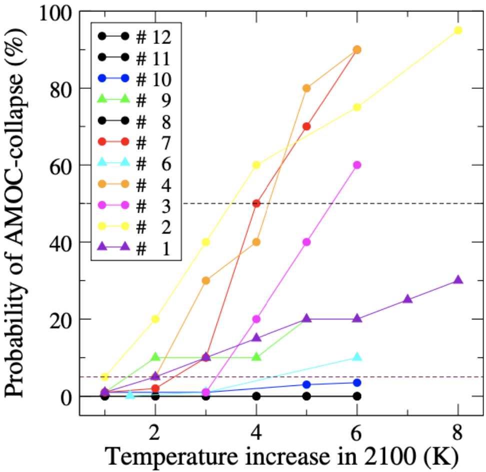
[2] K. Zickfeld, A. Levermann, M.G. Morgan, T. Kuhlbrodt, S. Rahmstorf, D.W. Keith, *Expert judgements on the response of the Atlantic meridional overturning circulation to climate change*, Climatic Change, 82(3), 235-265

[3] Note: although the link between cumulative emissions and temperature increase has been shown to be robustly described by a linear relationship (Matthews et al. 2009, IPCC 2013), the magnitude of the so-called carbon-climate response describing this relationship remains actually uncertain.

The following plot is taken from Zickfeld et al. It clearly shows a decrease in the disagreement (i.e., in the model uncertainty) for lower temperatures, that is, in our setting, for higher abatements.



Thank you for your attention!

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